Topic: Sum-to-product identities

Question: Choose the correct expression.

Express $\sin(8\theta) - \sin(20\theta)$ as the product of two trig functions and a negative number a.

Answer choices:

A $\sin(8\theta) - \sin(20\theta) = -4\cos(7\theta)\sin(3\theta)$

B $\sin(8\theta) - \sin(20\theta) = (-1)\cos(14\theta)\sin(6\theta)$

C $\sin(8\theta) - \sin(20\theta) = (-1)\cos(7\theta)\sin(3\theta)$

D $\sin(8\theta) - \sin(20\theta) = -2\cos(14\theta)\sin(6\theta)$

© Krista King Math

Solution: D

By one of the sum-to-product identities,

$$\sin(8\theta) - \sin(20\theta) = 2\cos\left(\frac{8\theta + 20\theta}{2}\right)\sin\left(\frac{8\theta - 20\theta}{2}\right)$$

Simplifying:

$$\sin(8\theta) - \sin(20\theta) = 2\cos\left(\frac{28\theta}{2}\right)\sin\left(\frac{-12\theta}{2}\right)$$

$$\sin(8\theta) - \sin(20\theta) = 2\cos(14\theta)\sin(-6\theta)$$

Using the odd identity for the sine function, we obtain

$$\sin(8\theta) - \sin(20\theta) = 2\cos(14\theta)\left[-\sin(6\theta)\right]$$

$$\sin(8\theta) - \sin(20\theta) = -2\cos(14\theta)\sin(6\theta)$$

This is in the prescribed form, with $a = -2$.

© Krista King Math

Topic: Sum-to-product identities

Question: Find the exact value.

$$\cos\frac{15\pi}{8}+\cos\frac{7\pi}{8}$$

Answer choices:

A -2

B 1

C 0

D 2

© Krista King Math

Solution: C

By one of the sum-to-product identities,

$$\cos\theta + \cos\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

Setting θ to $15\pi/8$ and α to $7\pi/8$, we have

$$\cos\left(\frac{15\pi}{8}\right) + \cos\left(\frac{7\pi}{8}\right) = 2\cos\left[\frac{\left(\frac{15\pi}{8}+\frac{7\pi}{8}\right)}{2}\right]\cos\left[\frac{\left(\frac{15\pi}{8}-\frac{7\pi}{8}\right)}{2}\right]$$

$$\cos\left(\frac{15\pi}{8}\right) + \cos\left(\frac{7\pi}{8}\right) = 2\cos\left[\frac{\left(\frac{22\pi}{8}\right)}{2}\right]\cos\left[\frac{\left(\frac{8\pi}{8}\right)}{2}\right]$$

Reducing the fractions within the numerators of the expressions in square brackets to lowest terms, we obtain

$$\cos\left(\frac{15\pi}{8}\right) + \cos\left(\frac{7\pi}{8}\right) = 2\cos\left[\frac{\left(\frac{11\pi}{4}\right)}{2}\right]\cos\left(\frac{\pi}{2}\right)$$

$$\cos\left(\frac{15\pi}{8}\right) + \cos\left(\frac{7\pi}{8}\right) = 2\cos\left(\frac{11\pi}{8}\right)\cos\left(\frac{\pi}{2}\right)$$

Since $\cos(\pi/2) = 0$, we find that

$$\cos\left(\frac{15\pi}{8}\right) + \cos\left(\frac{7\pi}{8}\right) = 2\cos\left(\frac{11\pi}{8}\right)(0) = 0$$

© Krista King Math

Topic: Sum-to-product identities

Question: Which of the following is a trigonometric identity?

Answer choices:

A $\sin\left(\theta + \frac{3\pi}{2}\right) = \frac{1}{2} - \sin\theta$

B $\sin\left(\theta + \frac{3\pi}{2}\right) = -2\cos\left(\theta - \frac{3\pi}{4}\right) - \sin\theta$

C $\sin\left(\theta + \frac{3\pi}{2}\right) = \left(1 - \frac{1}{\sqrt{2}}\right) - \sin\theta$

D $\sin\left(\theta + \frac{3\pi}{2}\right) = -\sqrt{2}\sin\left(\theta + \frac{3\pi}{4}\right) - \sin\theta$

© Krista King Math

Solution: D

We'll prove that

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = -\sqrt{2}\sin\left(\theta + \frac{3\pi}{4}\right)$$

is an identity. The identity

$$\sin\left(\theta + \frac{3\pi}{2}\right) = -\sqrt{2}\sin\left(\theta + \frac{3\pi}{4}\right) - \sin\theta$$

follows immediately from that.

By one of the sum-to-product identities,

$$\sin\theta + \sin\alpha = 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

Setting α to $\theta + (3\pi/2)$, we get

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = 2\sin\left[\frac{\theta + \left(\theta + \frac{3\pi}{2}\right)}{2}\right]\cos\left[\frac{\theta - \left(\theta + \frac{3\pi}{2}\right)}{2}\right]$$

Regrouping within the numerators of the expressions in square brackets:

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = 2\sin\left[\frac{(\theta+\theta) + \left(\frac{3\pi}{2}\right)}{2}\right]\cos\left[\frac{(\theta-\theta) - \left(\frac{3\pi}{2}\right)}{2}\right]$$

© Krista King Math

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = 2\sin\left[\frac{2\theta + \left(\frac{3\pi}{2}\right)}{2}\right]\cos\left[\frac{0 - \left(\frac{3\pi}{2}\right)}{2}\right]$$

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = 2\sin\left[\frac{2\theta + \left(\frac{3\pi}{2}\right)}{2}\right]\cos\left[\frac{-\left(\frac{3\pi}{2}\right)}{2}\right]$$

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = 2\sin\left(\theta + \frac{3\pi}{4}\right)\cos\left(-\frac{3\pi}{4}\right)$$

We know that $\cos(-3\pi/4) = -1/\sqrt{2}$, so

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = 2\sin\left(\theta + \frac{3\pi}{4}\right)\left(-\frac{1}{\sqrt{2}}\right)$$

Changing the order of the last two factors, we get

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = 2\left(-\frac{1}{\sqrt{2}}\right)\sin\left(\theta + \frac{3\pi}{4}\right)$$

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = -\sqrt{2}\sin\left(\theta + \frac{3\pi}{4}\right)$$

Note that the expression on the left-hand side of all four answer choices is the same. Therefore, to prove that answer choices A, B, and C are not identities, it suffices to exhibit one angle θ for which the value of the expression on the right-hand side of the identity in answer choice D is

© Krista King Math

unequal to the value of the expression on the right-hand side of each of the other three answer choices.

Well, let $\theta = 0$. Then the right-hand side of answer choice D is

$$-\sqrt{2}\sin\left(0+\frac{3\pi}{4}\right)-\sin(0)$$

$$-\sqrt{2}\sin\left(0+\frac{3\pi}{4}\right)-\sin(0)=-\sqrt{2}\sin\left(\frac{3\pi}{4}\right)-\sin(0)$$

Since $\sin(3\pi/4) = 1/\sqrt{2}$ and $\sin 0 = 0$,

$$-\sqrt{2}\sin\left(0+\frac{3\pi}{4}\right)-\sin(0)=-\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)-0=-1-0=-1$$

With $\theta = 0$, the right-hand side of answer choice A is

$$\frac{1}{2}-\sin(0)$$

Since $\sin 0 = 0$,

$$\frac{1}{2}-\sin(0)=\frac{1}{2}-0=\frac{1}{2}\neq -1$$

With $\theta = 0$, the right-hand side of answer choice B is

$$-2\cos\left(0-\frac{3\pi}{4}\right)-\sin(0)$$

$$-2\cos\left(0-\frac{3\pi}{4}\right)-\sin(0)=-2\cos\left(-\frac{3\pi}{4}\right)-\sin(0)$$

© Krista King Math

Since $\sin(-3\pi/4) = -1/\sqrt{2}$ and $\sin 0 = 0$,

$$-2\cos\left(0-\frac{3\pi}{4}\right)-\sin(0) = -2\left(-\frac{1}{\sqrt{2}}\right)-0 = \sqrt{2}-0 = \sqrt{2} \neq -1$$

With $\theta = 0$, the right-hand side of answer choice C is

$$\left(1-\frac{1}{\sqrt{2}}\right)-\sin(0)$$

Since $\sin 0 = 0$,

$$\left(1-\frac{1}{\sqrt{2}}\right)-\sin(0) = \left(1-\frac{1}{\sqrt{2}}\right)-0 = 1-\frac{1}{\sqrt{2}} \neq -1$$

© Krista King Math